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THE PATH OF LIGHT IN A GRAVITATIONAL FIELD.

By H. S. UHLER, Yale University.

While reading the recent books on the generalized principle of relativity, Einstein's theory, etc., I failed to find any numerical data which would give, even approximately, a quantitative idea of the angular deviation at various points on the curved ray which represents the course taken by a light wave in coming from a remote star to the earth after having passed close to the limb of the sun. All of the books examined give the total deviation,[1] $4m/R = 1.745''$, from infinity to infinity (that is, the angle between the asymptotes to the ray) but nothing more about numerical angles. Since my investigation of this problem may contain something of interest to the readers of the MONTHLY, it seems appropriate to present the details of the analysis in this place.

As a purely mathematical problem, we shall first find the polar equation of a ray of light in an unlimited medium in which the index of refraction is a linear function of the reciprocal of the distance of the wave from the pole or center of symmetry. We shall then simplify the equation by adapting it to the special case of a gravitational field. Finally, we shall obtain so much of a series expansion as will be necessary for the calculation of the very small angular deviations involved.

In order to avoid undue distraction in the midst of the subsequent analysis we shall now state a theorem,[2] borrowed from geometrical optics, which will constitute the keynote to the plan of attack to be followed in this paper. For a transparent medium having the property that the index of refraction possesses spherical symmetry with respect to a given center, the product of the index of refraction at any point on a ray of light in the medium by the length of the perpendicular dropped from the center of symmetry upon the tangent to the ray at the point in question is constant. Let $n_1, n_2, n_3, \cdots$ denote the indices of refraction at points along the same ray and let $p_1, p_2, p_3, \cdots$ symbolize the lengths of the corresponding perpendiculars, then

$$n_1 p_1 = n_2 p_2 = n_3 p_3 = \cdots = constant. \tag{1}$$

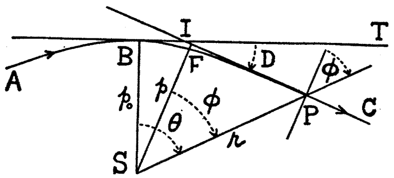
This fact is independent of the functional relation between the index of refraction and the distance from the center of symmetry.

In the figure let S denote the center of symmetry and let ABC represent the curved ray of light. B is the point on the ray nearest to S. Take the line SB as the axis of polar coördinates (r, θ) of any point P on the ray. $SP = r$, $\angle BSP = \theta$. Draw tangents BT and FP to the curved ray at the points B

[1] R denotes the radius of the sun and m a quantity which Eddington calls "the mass of the sun in astronomical units," while Weyl styles it the "gravitational radius." m is always expressed in terms of a unit of length, usually the kilometer.

[2] *Cf.* R. S. Heath, *Geometrical Optics*, 1887, p. 328.

and P, respectively. Let the lengths of the perpendiculars (SB and SF) dropped from S upon these tangent lines be denoted by p_0 and p. Let ϕ represent the acute angle which the radius vector to the point P makes with the normal to the ray at P. Then $\angle FSP = \phi$. The deviation of the ray, D, at the point P will be defined as the angle which the tangent FP makes with the apsidal tangent BT. $D = \angle TIP = \angle BSF = \theta - \phi$.



In general

$$\tan\phi = \frac{1}{r}\cdot\frac{dr}{d\theta}. \tag{2}$$

From the figure

$$\tan\phi = \frac{\overline{FP}}{\overline{SF}} = \frac{\sqrt{r^2 - p^2}}{p}. \tag{3}$$

Elimination of ϕ from relations (2) and (3) gives

$$\frac{dr}{d\theta} = \frac{r}{p}\sqrt{r^2 - p^2}. \tag{4}$$

In order to integrate equation (4), p must be given as a function of r. As stated in the second paragraph let

$$n = h + \frac{l}{r}. \tag{5}$$

For the point B we may write $n_0 = h + l/p_0$. Hence, according to lemma (1) we have $p = r(hp_0 + l)/(hr + l)$.

Substituting the last expression for p in equation (4) we obtain

$$d\theta = \frac{dr}{r\sqrt{\left(\dfrac{hr + l}{hp_0 + l}\right)^2 - 1}}. \tag{6}$$

Putting $u \equiv 1/r$ and $u_0 \equiv 1/p_0$ in equation (6) we find

$$\frac{(\sqrt{h})u_0 d\theta}{\sqrt{h + 2lu_0}} = \frac{-du}{\sqrt{1 - \left[\dfrac{(h + 2lu_0)u - lu_0^2}{u_0(h + lu_0)}\right]^2}}.$$

Substituting z for the expression between the brackets, the last equation reduces to the simple form

$$\frac{\sqrt{h(h + 2lu_0)}}{h + lu_0}\,d\theta = \frac{-dz}{\sqrt{1 - z^2}};$$

whence

$$\frac{\sqrt{h(h + 2lu_0)}}{h + lu_0}\,\theta = \cos^{-1} z + c.$$

When $\theta = 0$; $r = p_0$, $u = u_0$, and $z = 1$, therefore $c = 0$. Consequently the equation of the curved ray may be written as

$$\frac{1}{r} = \frac{1}{p_0(hp_0 + 2l)} \left\{ l + (hp_0 + l) \cos \left[\frac{\sqrt{hp_0(hp_0 + 2l)}}{hp_0 + l} \theta \right] \right\}. \tag{7}$$

We shall next apply equation (7) to the special case of the gravitational field associated with the sun. The point S of the figure may now be considered as situated at the center of this star.

According to Einstein's theory of gravitation the index of refraction at the point P is given by $n = 1 + 2m/r$ to the first order of approximation in m/r. As a matter of fact, m^2/r^2 would be negligible in the case of the sun since, for this mass, $m = 1.473$ km., $p_0 = 6964 \times 10^2$ km. (corresponding to the sun's limb) and $m^2/p_0^2 = 4.474 \times 10^{-12}$. Consequently we may obtain an approximate equation of the path of a wave of light in the sun's gravitational field by neglecting all powers of m/r higher than the first in equation (7), and by putting $h = 1$ and $l = 2m$. [*Vide* relation (5).]

Under these conditions equation (7) reduces to

$$\frac{1}{r} = \frac{2m}{p_0^2} + \frac{p_0 - 2m}{p_0^2} \cos \theta. \tag{8}$$

This is the familiar equation of one branch of an hyperbola referred to the nearer focus as pole. The eccentricity of the hyperbola is given by $e = p_0/2m - 1$ which, in the practical case under consideration, has the very large value 2364×10^2. The transverse semi-axis $= (2mp_0)/(p_0 - 4m) = 2.946$ km., approximately.

For an infinite value of r equation (8) shows that θ is obtuse and that the approximate value of the total angular deviation, $(2D_\infty)$, from asymptote to asymptote, equals $4m/p_0$. (*Vide supra.*)

In order to calculate the deviations at various points along the curved ray it will be necessary to derive a special formula, since the numerical values of the deviations are so extremely small as to preclude the use of equation (8) in conjunction with ordinary logarithmic-trigonometric tables.

This may be accomplished in the following manner. Let

$$a \equiv \frac{p}{r} = \frac{p_0 + 2m}{r + 2m}, \qquad b \equiv \frac{p_0^2 - 2mr}{r(p_0 - 2m)}.$$

By referring to the figure and equation (8) we readily see that

$$D = \theta - \phi = \cos^{-1} b - \cos^{-1} a,$$

or

$$D = \sin^{-1} a - \sin^{-1} b,$$

or

$$D = \sin^{-1} [a \sqrt{1 - b^2} - b \sqrt{1 - a^2}]. \tag{9}$$

To the first order in m/p_0 and m/r we obtain

$$a \sqrt{1 - b^2} = c \sqrt{1 - \frac{p_0^2}{r^2}}, \qquad b \sqrt{1 - a^2} = \left(c - \frac{2m}{p_0}\right) \sqrt{1 - \frac{p_0^2}{r^2}},$$

where

$$c \equiv \frac{p_0 + 2m}{r} - \frac{2mp_0^2}{r^2(r + p_0)}.$$

Hence, relation (9) reduces to

$$D = \frac{2m}{p_0}\sqrt{1 - \left(\frac{p_0}{r}\right)^2} = \frac{1296 \times 10^3\, m}{\pi p_0 r}\sqrt{(r - p_0)(r + p_0)}''. \tag{10}$$

The first form of relation (10) shows that the deviation very rapidly approaches the asymptotic limit $2m/p_0$ as r increases, and the second form is especially convenient for logarithmic computation.

In calculating the data collected in the table given below the values 1.473 km. and 6964×10^2 km. were taken for m and p_0, respectively. Thus the table pertains to a ray that has grazed the sun's limb. For integral values, θ was taken as the independent variable and r was calculated from equation (8). The numbers given for r in the table have been rounded off. The angles tabulated in the right hand column were obtained by the aid of formula (10).

$\theta°$	r km.	D''
0	6964×10^2	0
5	6991×10^2	0.07605
10	7071×10^2	0.15152
15	7210×10^2	0.22584
20	7411×10^2	$0.2984\bar{4}$
25	7684×10^2	0.36876
30	8041×10^2	0.43628
35	8501×10^2	0.50048
40	9091×10^2	0.56087
45	9849×10^2	0.61700
50	1083×10^3	0.66842
55	1214×10^3	0.71476
60	1393×10^3	0.75566
65	1648×10^3	0.79081
70	2036×10^3	0.81994
75	2691×10^3	0.84283
80	4010×10^3	0.85931
85	7990×10^3	0.86925
θ_M	5794×10^4	0.87250
θ_E	1495×10^5	0.87256
90	1646×10^8	0.87257
θ_∞	∞	0.87257

The fourth and third rows from the bottom of the table pertain respectively to the orbits of the planets Mercury and Earth. $\theta_M = 89° 18' 41.5''$, $\theta_E = 89°$

44′ 0″, and $\theta_\infty = 90° + D_\infty$. The data in the last four rows fulfill the expectation that the curved ray has sensibly reached its asymptote not only for the earth's orbit but even for an observer at the distance of the inferior planet.[1]

GRAPHICAL DISCUSSION OF THE ROOTS OF A QUARTIC EQUATION.

By E. L. REES, University of Kentucky.

It is the purpose of this note to give a graphical study of the conditions which determine the nature of the roots of a quartic equation. Using the reduced form $f(x) = x^4 + qx^2 + rx + s = 0$, with q, r and s real and with discriminant Δ, we have types of the quartic for which the following are criteria[2] regarding the nature of the roots:

$\Delta < 0$, roots distinct, two real, two imaginary;

$\Delta > 0$, roots distinct, all real or all imaginary;

$$q < 0,\ s > \frac{q^2}{4}, \text{ roots imaginary;}$$

$$s < \frac{q^2}{4}, \text{ roots real;}$$

$$q \geqq 0, \text{ roots imaginary;}$$

$\Delta = 0$, at least two equal roots;

$$q < 0,\ s > \frac{q^2}{4}, \text{ two equal real roots, two imaginary;}$$

$$-\frac{q^2}{12} < s < \frac{q^2}{4}, \text{ roots real, two and only two equal;}$$

$$s = \frac{q^2}{4}, \text{ two pairs of equal real roots;}$$

$$s = -\frac{q^2}{12}, \text{ roots real, three equal;}$$

$$q > 0,\ s > 0,\ r \neq 0, \text{ two equal real roots, two imaginary;}$$

$$s = \frac{q^2}{4},\ r = 0, \text{ two pairs of equal imaginary roots;}$$

$$s = 0, \text{ two equal real roots, two imaginary;}$$

$$q = 0,\ s > 0, \text{ two equal real roots, two imaginary;}$$

$$s = 0, \text{ four equal real roots.}$$

The discriminant is the product of the squares of the differences of the roots

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[1] The deviation of a ray from its original direction at the star is obtained by adding $2m/p_0$ to the value of D given in the table, and the total deviation, 1.745″, from asymptote to asymptote is twice the value of D given in the last two lines of the table.—EDITORS.

[2] Compare L. E. Dickson, *Elementary Theory of Equations*, 1914, p. 45.